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THE FREEBOUNDARY IN A MINIMIZATION PROBLEM

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1. Introduction

In this paper we study a minimization problem

$$\min I^{\lambda}(u)=\min\int_{\Omega}\frac{|\nabla u|^{p}}{p}+\frac{\lambda}{\gamma+1}u^{\gamma+1}\,dx,\quad p\geq 2,\quad \gamma\in[0,p-1)$$

with respect to $K=W_0^{1,p}(\Omega)+u_0$, where λ is a positive constant. Here we consider the case boundary data u_0 is constant, say, $u_0=1$. The motivation of this problem comes from reaction diffusion models. We refer various references in [6] and [8] for practical motivations.

From variational principle we note that the minimizer satisfies the Euler-Lagrange equation

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)=\lambda u^{\gamma}\quad \text{in } \Omega.$$

In fact the existence and uniqueness follows from convexity of the functional I^{λ} on $W_0^{1,p}+u_0$. An interesting fact is that if $\gamma<p-1$, then there appears deadcore $N_{\lambda}(u)=\{x\in\Omega: u(x)=0\}$. Here we call $F(u)=\partial\{u>0\}$ the free boundary.

We shall study the nature of free boundary and deadcore. Our main result is that if $\partial\Omega$ has positive mean curvature, then the smooth portion of free boundary has also positive mean curvature. Hence in two dimensional case if Ω is convex, then the deadcore is also convex. Friedman and Phillips[8] considered the case when $p=2$. Moreover the convexity of the graph of the solutions to various minimization problems were considered by many authors([4], [10]).

We also study the asymptotic behaviour of free boundary with respect to λ. Indeed for two dimensional case van Duijn and Peletier[7] studied the behaviour of free boundary for discontinuous boundary data.

We assume $\partial\Omega$ is smooth and use the following symbol, $B_R(x_0) = \{x : |x - x_0| < R\}$.

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2. Asymptotic behavior of deadcore as $\lambda \to \infty$

In this section we study the asymptotic behavior of u_λ as λ goes to ∞. First we prove that u_λ decreases at each point as $\lambda \to \infty$. This follows from standard comparison method.

Lemma 2.1. *Let $0 < \lambda_2 < \lambda_1$, then $u_{\lambda_2} < u_{\lambda_1}$ on $\{x \in \Omega : u_{\lambda_2}(x) > 0\}$.*

Proof. We regularize I^λ by

$$\int_\Omega \frac{1}{p}\left(\varepsilon + |\nabla u|^2\right)^{\frac{p}{2}} + \frac{\lambda}{\gamma+1} u^{\gamma+1}\, dx, \quad u = 1 \text{ on } \partial\Omega$$

and let u_λ^ε be the minimizer. Then $u_\lambda^\varepsilon \in C^{2,\alpha}(\Omega)$ for all $0 < \alpha < 1$. If $w(x) = u_{\lambda_1}^\varepsilon(x) - u_{\lambda_2}^\varepsilon(x)$ attains a positive maximum at $x_0 \in \Omega$, then

$$\begin{aligned}
0 \geq& \operatorname{div}\left((\varepsilon + |\nabla u_{\lambda_1}^\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_{\lambda_1}^\varepsilon - (\varepsilon + |\nabla u_{\lambda_2}^\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_{\lambda_2}^\varepsilon\right) \\
=& \lambda_1 \left(u_{\lambda_1}^\varepsilon(x_0)\right)^\gamma - \lambda_2 \left(u_{\lambda_2}^\varepsilon(x_0)\right)^\gamma \\
\geq& (\lambda_1 - \lambda_2) u_{\lambda_2}^{\varepsilon,\gamma}(x_0) > 0.
\end{aligned}$$

Note that $\nabla u_{\lambda_1}^\varepsilon(x_0) = \nabla u_{\lambda_2}^\varepsilon(x_0)$. Hence we get $a_{ij} w_{ij}^\varepsilon > 0$ for

$$a_{ij} = \left(\varepsilon + |\nabla u_{\lambda_2}^\varepsilon|^2\right)^{\frac{p-2}{2}} \left(\delta_{ij} + \frac{u_{\lambda_2,x_i}^\varepsilon u_{\lambda_2,x_j}^\varepsilon}{\varepsilon + |\nabla u_{\lambda_2}^\varepsilon|^2}\right)$$

and this contradicts to the assumption w attains maximum at x_0. □

Consequently we have

$$N_{\lambda_1} \subset \operatorname{int} N_{\lambda_2} \quad \text{if } 0 < \lambda_1 < \lambda_2.$$

The following theorem is our main result in this section and the case when $p = 2$ was considered by Friedman and Phillips[8].

We define $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$.

Theorem 2.2. *There exist positive constants a, c and λ_0 depending only on n, p and γ such that*

$$\Omega_{a/\sqrt[p]{\lambda}+c/(\sqrt[p]{\lambda})^2} \subset N_\lambda \subset \Omega_{a/\sqrt[p]{\lambda}-c/(\sqrt[p]{\lambda})^2}$$

for all $\lambda > \lambda_0$.

Proof. We let $w_\lambda(x) = u_\lambda\left(\frac{x}{\sqrt[p]{\lambda}}\right)$, then

$$\operatorname{div}(|\nabla w_\lambda|^{p-2}\nabla w_\lambda) = w^\gamma.$$

Hence from elliptic estimate

$$|\nabla w| \leq C$$

since $|w| = |u| \leq 1$. Hence we get $|\nabla u| \leq c\sqrt[p]{\lambda}$ and

$$N_\lambda \subset \Omega_{c/\sqrt[p]{\lambda}}.$$

On the other hand if we set $v(x) = A|x - x_0|^{\frac{p}{p-1-\gamma}}$, then

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = C_0^p A^{p-1-\gamma} v^\gamma,$$

where $C_0 = \dfrac{(p^{p-1}(\gamma+1)(p-1))^{\frac{1}{p}}}{p-1-\gamma}$. We take A satisfying $Ad^{\frac{p}{p-1-\gamma}} = 1$, where $d = \operatorname{dist}(x_0, \partial\Omega)$, then $v \geq 1$ on $\partial\Omega$. If $C_0^p A^{p-1-\gamma} = \lambda$, that is, $d = \dfrac{C_0}{\sqrt[p]{\lambda}}$, then $v \geq u$ and $v(x_0) = u(x_0) = 0$. This implies

$$\Omega_{C_0/\sqrt[p]{\lambda}} \subset N_\lambda \subset \Omega_{C/\sqrt[p]{\lambda}}. \tag{1}$$

Now we refine the previous estimates. Let $y \in \partial\Omega$ and $B_R \subset \Omega$ such that $y \in \partial B_R$. Let U be the radial minimizer if I^λ, then $u^\lambda \leq U$ and $U'(r) \geq 0$. U satisfies

$$(p-1)|U'|^{p-2}U'' + \frac{n-1}{r}|U'|^{p-2}U' = \lambda U^\gamma$$

and

$$Z(s) = U\left(R - \frac{\gamma_0}{\sqrt[p]{\lambda}} + \frac{s}{\sqrt[p]{\lambda}}\right) \quad (\ \gamma_0 \text{ is to be determined})$$

satisfies

$$(p-1)|Z'|^{p-2}Z'' + \frac{n-1}{\rho\sqrt[p]{\lambda}+s}|Z'|^{p-2}Z' = Z^\gamma, \tag{2}$$

where $\rho = R - \frac{\gamma_0}{\sqrt[p]{\lambda}}$.

From (1) $\gamma_0 \leq C$ independent λ. Multiplying both side of (2) by $Z'(s)$, we get

$$\frac{p-1}{p}(|Z'|^p)' + \frac{n-1}{\rho\sqrt[p]{\lambda}+s}|Z'|^p = Z^\gamma Z'.$$

Hence we obtain

$$(|Z'|^p)' + \frac{(n-1)p}{p-1}\frac{1}{\rho\sqrt[p]{\lambda}+s}|Z'|^p = \frac{p}{(p-1)(\gamma+1)}(Z^{\gamma+1})'$$

and

$$(|Z'|^p)' + \frac{C}{\sqrt[p]{\lambda}}|Z'|^p \geq \frac{p}{(p-1)(\gamma+1)}(Z^{\gamma+1})'$$

for some C. From this we obtain

$$\left(e^{Cs/\sqrt[p]{\lambda}}|Z'|^p\right)' \geq \frac{p}{(p-1)(\gamma+1)}e^{Cs/\sqrt[p]{\lambda}}(Z^{\gamma+1})'$$

and

$$\begin{aligned}
|Z'|^p(s) \geq& e^{-Cs/\sqrt[p]{\lambda}}\frac{p}{(p-1)(\gamma+1)}\int_0^s e^{Ct/\sqrt[p]{\lambda}}(Z^{\gamma+1})'\,dt \\
=&\frac{p}{(p-1)(\gamma+1)}Z^{\gamma+1}(s) - \frac{p}{(p-1)(\gamma+1)}\frac{C}{\sqrt[p]{\lambda}}\int_0^s e^{-C(s-t)/\sqrt[p]{\lambda}}Z^{\gamma+1}\,dt.
\end{aligned}$$

Recalling that $Z'(t) \geq 0$ we get

$$|Z'|^p(s) \geq \frac{p}{(p-1)(\gamma+1)}\left(1 - \frac{C}{\sqrt[p]{\lambda}}\right)Z^{\gamma+1}(s).$$

On the other hand

$$\begin{cases} \eta'(s) &= \left(\frac{p}{(p-1)(\gamma+1)}\eta^{\gamma+1}(s)\right)^{\frac{1}{p}} \\ \eta(0) &= 1 \end{cases}$$

has a unique solution as long as $\eta > 0$. It determines a unique number $a > 0$ such that

$$\eta(-a) = 0.$$

Letting $\zeta(s) = \eta(-a+s)$ we have

$$\begin{cases} \zeta'(s) & = & \left(\dfrac{p}{(p-1)(\gamma+1)}\zeta^{\gamma+1}(s)\right)^{\frac{1}{p}} & \text{for } 0 < s < a \\ \zeta(s) & > & 0 & \text{for } 0 < s < a \\ \zeta(0) & = & 0 & \\ \zeta(a) & = & 1. & \end{cases}$$

The function

$$\tilde{\zeta}(s) = \zeta\left(s\left(1 - \frac{C}{\sqrt[p]{\lambda}}\right)^{\frac{1}{p}}\right)$$

satisfies

$$\left(\tilde{\zeta}(s)\right)' = \left(1 - \frac{C}{\sqrt[p]{\lambda}}\right)^{\frac{1}{p}} \left(\frac{p}{(p-1)(\gamma+1)}\tilde{\zeta}^{\gamma+1}\right)^{\frac{1}{p}}.$$

By comparison we also have

$$Z(s) \geq \tilde{\zeta}(s) = \zeta\left(s\left(1 - \frac{C}{\sqrt[p]{\lambda}}\right)^{\frac{1}{p}}\right).$$

Since $U(R) = 1$ implies $Z(\gamma_0) = 1$, we conclude that

$$\gamma_0\left(1 - \frac{C}{\sqrt[p]{\lambda}}\right)^{\frac{1}{p}} \leq a.$$

Recalling that

$$u_\lambda \leq U,$$

we deduce that

$$u_\lambda(|x - x_0|) \leq Z(|x - x_0|) \leq U\left(R - \frac{\gamma_0}{\sqrt[p]{\lambda}} + \frac{|x - x_0|}{\sqrt[p]{\lambda}}\right)$$

and $U(x_0) = 0$ implies

$$N_\lambda \supset \Omega_{R-\gamma_0/\sqrt[p]{\lambda}} \supset B_{R-a/\sqrt[p]{\lambda}-C/\lambda^{\frac{2}{p}}}.$$

This completes the first part of the theorem.

To prove the second part we let v be the radial solution of

$$\begin{cases} \operatorname{div}(|\nabla v|^{p-2}\nabla v) = \lambda v^\gamma & \text{in} & B_{R_1} \setminus B_{R_0} \\ v = 1 & \text{on} & \partial B_{R_0} \\ v = 0 & \text{on} & \partial B_{R_1}, \end{cases}$$

where $B_{R_1} \supset \Omega$ and $\bar{B}_{R_0} \cap \Omega = \{y\}$ for some y. Then from comparison $v \leq u_\lambda$ and $v'(r) \leq 0$. Then considering $\bar{Z}(s) = V\left(R + \frac{\bar{\gamma}}{\sqrt[p]{\lambda}} - \frac{s}{\sqrt[p]{\lambda}}\right)$ as in the proof of the first part, we prove the second part. □

3. Convexity of deadcore

The following maximum principle for polynomial growth case is relatively well known(see Chapter 7 in [13]).

Lemma 3.1. *Let Ω be a bounded regular ($\partial\Omega \in C^2$) open set. Then if $\partial\Omega$ has nonnegative mean curvature, then for every $x \in \bar{\Omega}$*

$$|\nabla u(x)|^p \leq \frac{p}{p-1}\frac{\lambda}{\gamma+1}\left(u^{\gamma+1}(x) - m^{\gamma+1}\right),$$

where $m = \min_{x\in\bar{\Omega}} u(x)$.

Corollary 3.2. *Let Ω be convex domain in $\mathbf{R}^n$ and let x_m be the point at which the minimum $u(x_m) = m \geq 0$ occurs. Then*

$$\operatorname{dist}(x_m, \partial\Omega) \geq \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_m^1 \left(\frac{\lambda}{\gamma+1}\left(s^{\gamma+1} - m^{\gamma+1}\right)\right)^{-\frac{1}{p}} ds$$

In particular the null set N is empty if

$$\rho < \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^1 \left(\frac{\lambda}{\gamma+1}\right)^{-\frac{1}{p}} s^{-\frac{\gamma+1}{p}}.$$

Proof. Let $x_1 \in \partial\Omega$ and let r be the arc length on straight segment joining x_m to x_1. Let x_2 be a point in this segment such that $u(x_2) = m$ and $u(x) > m$ for all x between x_2 and x_1.

Then

$$\frac{du}{dr} \leq |\nabla u| \leq \left(\frac{p}{p-1}\right)^{\frac{1}{p}} \left(\int_m^u f(t)\,dt\right)^{\frac{1}{p}}.$$

So

$$\frac{dr}{du} \geq \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \frac{1}{\left(\int_m^u f(t)\,dt\right)^{\frac{1}{p}}}$$

and integrating from x_2 to x_1,

$$\begin{aligned}\operatorname{dist}(x_m, x_1) &\geq \operatorname{dist}(x_2, x_1)\\ &\geq \left(\frac{p-1}{p}\frac{\gamma+1}{\lambda}\right)^{\frac{1}{p}} \int_m^1 \frac{ds}{(s^{\gamma+1}-m^{\gamma+1})^{\frac{1}{p}}}.\end{aligned}$$

□

We let

$$\psi(u) = \left(\frac{p-1}{p}\frac{\gamma+1}{\lambda}\right) \frac{p}{p-\gamma+1} u^{\frac{p-\gamma-1}{p}},$$

then from the Hausdorff measure estimate of free boundary[5] we have

$$\operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi) = d\Lambda + I_{\{u>0\}} C\psi^{-1}(1-|\nabla\psi|^p),$$

where $d\Lambda = d\mathcal{H}^{n-1}F_{\text{reg}}(u) + \theta(x)d\mathcal{H}^{n-1}F_{\text{sing}}(u)$ and C depends only on n, p, γ, θ bounded. Here I is the usual characteristic function. Moreover

$$\psi^{-1}(1-|\nabla\psi|^p) \in L^1_{\text{loc}}.$$

From Green's formula we note that if D is a subdomain of Ω with piecewise smooth boundary ∂D and with $\mathcal{H}^{n-1}(F(u)\cap\partial D) = 0$, then

$$\int_D \operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi)\, dx = \int_{\partial D\cap\{u>0\}} |\nabla\psi|^{p-2}\nabla\psi\cdot\nu\, d\mathcal{H}^{n-1}.$$

Hence from the above observation if D has piecewise smooth boundary and $\mathcal{H}^{n-1}(F(u)\cap \partial D) = 0$, then

$$\begin{aligned}\int_{D\cap F_{\text{red}}} d\mathcal{H}^{n-1} + \int_{D\cap F_{\text{sing}}} \theta d\mathcal{H}^{n-1} &= -\int_{D\cap\{u>0\}} \operatorname{div}(|\nabla\psi|^{p-2}\nabla\psi)\, dx\\ &\quad + \int_{\partial D\cap\{u>0\}} |\nabla\psi|^{p-2}\nabla\psi\cdot\nu\, d\mathcal{H}^{n-1}.\end{aligned}$$

Therefore with the argument by Friedman and Phillips(see Theorem 4.3 in [8]) we prove the following Corollary.

Corollary 3.3. *Every C^2 portion of $F(u)$ has nonnegative mean curvature.*

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